WEEK 13:

$A \subseteq B$ open subring $\quad \mathrm{Cont}(B)_{an} \xrightarrow{f} \mathrm{Cont}(A)_{an}$ we didn't show that f is surjective.

Example: (not true without analytic) cont. valuations of $\mathbb{Z}_p\langle x\rangle$ don't all extend to $\mathbb{Q}_p\langle x\rangle$.

Proof of f surjective: Let $|\cdot| \in \mathrm{Cont}(A)_{an}$ and let $\mathrm{Supp}\,|\cdot| = p = A$.
$|\cdot|$ analytic $\Rightarrow p \not\supseteq I$ where I is an ideal of definition $\Rightarrow p \not\supseteq A^{\circ\circ}$ as $I \subseteq A^{\circ\circ}$. Let $s \in A^{\circ\circ} \setminus p$. Notice that for any element $b \in B$, $\exists n > 0$ st $I^n . b \subseteq A$. As $s \in A^{\circ\circ}$ $\exists j$ st $s^j \in I^n$ therefore $s^j . b \in A$.
Therefore,

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] \\ A_s \arrow[r, "\cong"] & B_s \end{tikzcd} \qquad \begin{tikzcd} p \arrow[d] & q \\ p.A_s \arrow[r, "\cong"] & pB_s \arrow[u] \end{tikzcd}$$

and $A_s \cong B_s$ let $q = B \cap p.B_s$. Note that $q \cap B = p$.

Consider the following diagram:

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & A_s \arrow[dr, "\cong"] \arrow[dd] & \\
& B \arrow[rr] \arrow[dd] & & B_s \arrow[dd] \\
A/p \arrow[rr] \arrow[dr] & & (A/p)_s \arrow[dr, "\cong"] \arrow[dd] & \\
& B/q \arrow[rr] & & (B/q)_s \arrow[dd] \\
& & \mathrm{Frac}(A/p) \arrow[dr, "\cong"] & \\
& & & \mathrm{Frac}(B/q)
\end{tikzcd}$$

using the isomorphism $\mathrm{Frac}(A/p) \cong \mathrm{Frac}(B/q)$ we can extend $|\cdot|$ to B.

Recall: (A, A^+) is perfectoid affinoid if
- affinoid
- A complete & $\underset{A^\circ \text{ bounded}}{\text{uniform}}$ $\Rightarrow A^\circ = A_0$ can be taken.
- $\exists \varpi \in A$ pseudo-uniformizer $[\varpi \in A^{\circ\circ}, \varpi \in A^\times]$ such that
  (i) $\varpi^p | p$ in A°. $\quad$ (ii) $A^\circ/\varpi \xrightarrow{\cong} A^\circ/\varpi^p$ is an isomorphism.
  $\qquad x \longmapsto x^p$

Example: • $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}^p$ can do the same with $\mathbb{Q}_p$ to get (K, K^+) • $\widehat{\mathbb{Z}_p[p^{1/p^\infty}, x^{1/p^\infty}]}^p$ can do the same with $\mathbb{Q}$ to get (A, A^+)
We have $\mathrm{Spa}(A, A^+) = \varprojlim_{x \mapsto x^p} \mathrm{Spa}(K\langle x\rangle, K^+\langle x\rangle)$.

Proposition: A Tate ring, $A_0 \subseteq A$ ring of definition, ϖ pseudo-unif.
(1) If $\varpi \in A_0 \Rightarrow \varpi . A_0$ is an ideal of def.
(2) A Hausdorff + uniform $\Rightarrow A$ is reduced.
(3) If $\varpi \in A_0$ then $A = A_0[\varpi^{-1}]$.

Proof:
(1) $\varpi . A_0$ open as $\varpi \in A^\times$. Fix $0 \in U$ open $\overset{A_0 \text{ bdd}}{\Rightarrow} \exists V \ni 0$ open : $V . A_0 \subseteq U$
as ϖ top. nilpotent $\exists n > 0$ st $\varpi^n \in V \Rightarrow \varpi^n . A_0 \subseteq U$.
(2) a nilpotent and fix $n > 0$. It is enough to show that $a \in \varpi^n . A^\circ$. This is equivalent to $\varpi^{-n} . a \in A^\circ$ but this follows as a is nilpotent.
(3) $\varpi \in A^{\circ\circ}$, $a \in A \Rightarrow \exists n > 0$ st $\varpi^n . a \in A_0$.

Proposition: A Tate, ϖ pseudo-unif. $\varpi^p | p$ in A° then
(1) $A^\circ/\varpi \longrightarrow A^\circ/\varpi^p$ inj.

(2) TFAE

(a) $A^\circ/pA^\circ \xrightarrow{F} A^\circ/pA^\circ$ is surj. (b) $A^\circ/\varpi A^\circ \longrightarrow A^\circ/\varpi^p A^\circ$ is surj.

Proof: (1) $a \in A^\circ$ st $a^p \in \varpi^p . A^\circ \Rightarrow (a.\varpi^{-1})^p \in A^\circ \Rightarrow a.\varpi^{-1} \in A^\circ \Rightarrow a \in \varpi A^\circ$
(2) Notice that (a) $\Rightarrow$ (b) follows from $\varpi^p | p$. (b) $\Rightarrow$ (a): Let $b_0 \in A^\circ$ we want $b_0 = b^p \bmod p$ we get b by approximation. By (b) $b_0 = a_0^p + \varpi^p . b_1$ $b_1 \in A_0$.
repeat w/ $b_1 \ldots \Rightarrow b_0 = \sum_{i \geq 0} a_i^p . \varpi^{pi} = (\sum_{i \geq 0} a_i w i)^p \bmod p$

Proposition: (A, A^+) perf. affinoid, $\varpi \in A^\circ$ ps.-unif. $\varpi | p$

$$\underset{x \mapsto x^p}{\varprojlim} A^\circ \xrightarrow{\cong} \underset{F}{\varprojlim} \underbrace{A^\circ/\varpi A^\circ}_{\text{positive char ring.}}$$ We denote $(A^\circ)^\flat = \underset{F}{\varprojlim} A^\circ/\varpi A^\circ$

topological monoids.

multiplicative not ring.

Proof: Need an inverse.
$([a_i])_{i \geq 0} \in \varprojlim A^\circ/\varpi A^\circ$ $\quad a_i^p = a_{i-1} \bmod \varpi$

Claim: $a_{i+1}^{p^{i+1}} = a_i^{p^i} \bmod \varpi^{i+1}$

Proof: $a_{i+1}^p = a_i \bmod \varpi \Rightarrow a_{i+1}^p = a_i + \varpi \overset{A_0}{\overset{\in}{c}}$, $(a_i + \varpi c)^p = a_i^p + \underbrace{\sum_{j>0} \binom{p}{j} \varpi^j . c^j . a_i^{p-j}}_{\text{divisible by } \varpi^2}$
and repeating this we obtain the claim.

$([a_i])_{i \geq 0} \longrightarrow (\lim_{j \to \infty} a_j^{p^j}, \lim_{j \to \infty} a_{j+1}^{p^j}, \_\_)$ is the inverse.
Moreover we have a map $(A^\circ)^\flat \xrightarrow{\#} A^\circ$ given by $([a_i]_{i \geq 0}) \longmapsto \lim_{j \to \infty} a_j^{p^j}$. $\square$

Addition on $\underset{x \mapsto x^p}{\varprojlim} A^\circ$: $(a_i)_{i \geq 0} + (b_i)_{i \geq 0} = (\lim_{j \to \infty} (a_j + b_j)^{p^j}, \lim_{j \to \infty} (a_{j+1} + b_{j+1})^{p^j}, \_)$
Can extend this to A.
One can show that $(A^\flat, (A^+)^\flat)$ is perfect affinoid of char p.